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THE INFLUENCE OF THE TESTING MACHINE ON THE BUCKLING OF CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION

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Abstract—A series of experiments has been carried out on electroformed cylindrical shells under axial compression to determine the effect of the stiffness of the testing machine on the buckling load. The effect of the testing machine has also been calculated using Tsien's criteria. It is shown that the calculated energy loads have a strong dependence on the testing machine while the experimental data are virtually independent of the testing machine stiffness.

NOTATION

E	Young's modulus
K	testing machine stiffness (lb/in)
K_s	shell stiffness, $2\pi RtE/L$
L	shell length
p	load applied to shell
p_{cl}	$2\pi Et^2/\sqrt{[3(1-\nu^2)]}$
PE	potential energy
PE_s	potential energy of the shell
R	shell radius
t	shell thickness
U_s	strain energy of the shell
δ	end shortening of shell
δ_{cl}	$tL/R\sqrt{[3(1-\nu^2)]}$
Δ	displacement of loading system
ν	Poisson's ratio
σ	$p/2\pi Rt$
σ_{cl}	$Et/R\sqrt{[3(1-\nu^2)]}$

INTRODUCTION

ANALYTICAL and experimental investigations on the effect of the method of load application on the buckling load of shells or other types of structures have been carried out by several authors [1–8]. In particular, the effect that is of interest here is the stiffness of the testing machine and how this enters into the determination of the collapse load. For shell structures which have a load deflection curve of the type shown in Fig. 1, Thompson [1] has shown that for infinitesimal disturbances, the effect of the loading apparatus does not enter into the determination of the collapse load. This results from the fact that the initial post buckling state is unstable under all loading conditions. The collapse load and buckling load are the same for this type of structure since there is a large drop off of load after buckling.

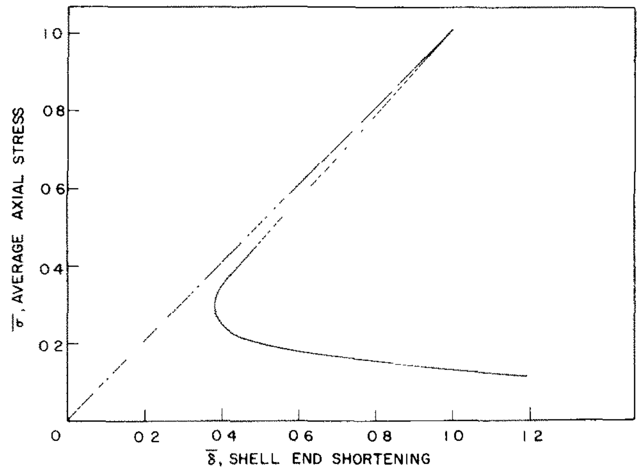

FIG. 1. Load vs. deflection for a cylindrical shell under axial compression.

If one considers the effect of finite disturbances on the buckling of shells, the nature of the loading system will be of importance. Unfortunately, the analysis for this type of loading does not exist except for very simple systems with few degrees of freedom. However, the effect of finite disturbances can be assessed very easily if one is willing to use some type of energy criterion. Once the characteristics of the loading apparatus and the loaded shell are known, the buckling load can be calculated. This will be done for two energy criteria advanced by Tsien [2,3] and the results compared with the experimental data.

Experimental investigations to determine the effect of the stiffness of the testing machine on the buckling load have been carried out using spherical caps [4], hemispherical shells [5], and cylindrical shells [6–8]. All of the investigators state that they have made tests in machines that are "soft", "intermediate", "hard", "dead weight" or "rigid". However, very little information is available to determine the rigidities of the various testing devices.

The work of Mossakovskii and Smelyi [6] involved the testing of 34 cylindrical shells in three machines. The results showed a difference in the buckling stress of about 0·04 Et/R, or about 12·5 per cent of the higher value, between the "soft" machine and the "hard" machine. This was compared with the results of Tsien's energy criterion [3] and the conclusion was reached that the criterion was valid and gave the proper influence of the testing machine. The comparison was made using a load deflection curve that gave a difference of "20 to 25 per cent", in buckling stress, for the extreme cases of rigid and dead loading. However, a recent calculation of the load deflection curve for a cylindrical shell[9] has shown that the difference between the two types of loading should be about 0·16 Et/R. In other words, the difference found by Mossakovskii and Smelyi is about 25 per cent of the difference that should be expected if this energy criterion is valid. In addition, the rigidities of the testing machines were not determined.

Almroth, Holmes and Brush [7] carried out several tests using high quality cylindrical shells. The testing device was a lever type of machine that could be operated either as a "dead weight" or "rigid" machine. The conclusion reached by the authors was that it made no difference if the loading was rigid or dead. They reached this by testing the same shell using both types of loading. However, due to the large amount of inertia associated with the poise and lever type of dead loading, it is extremely doubtful if the loading is

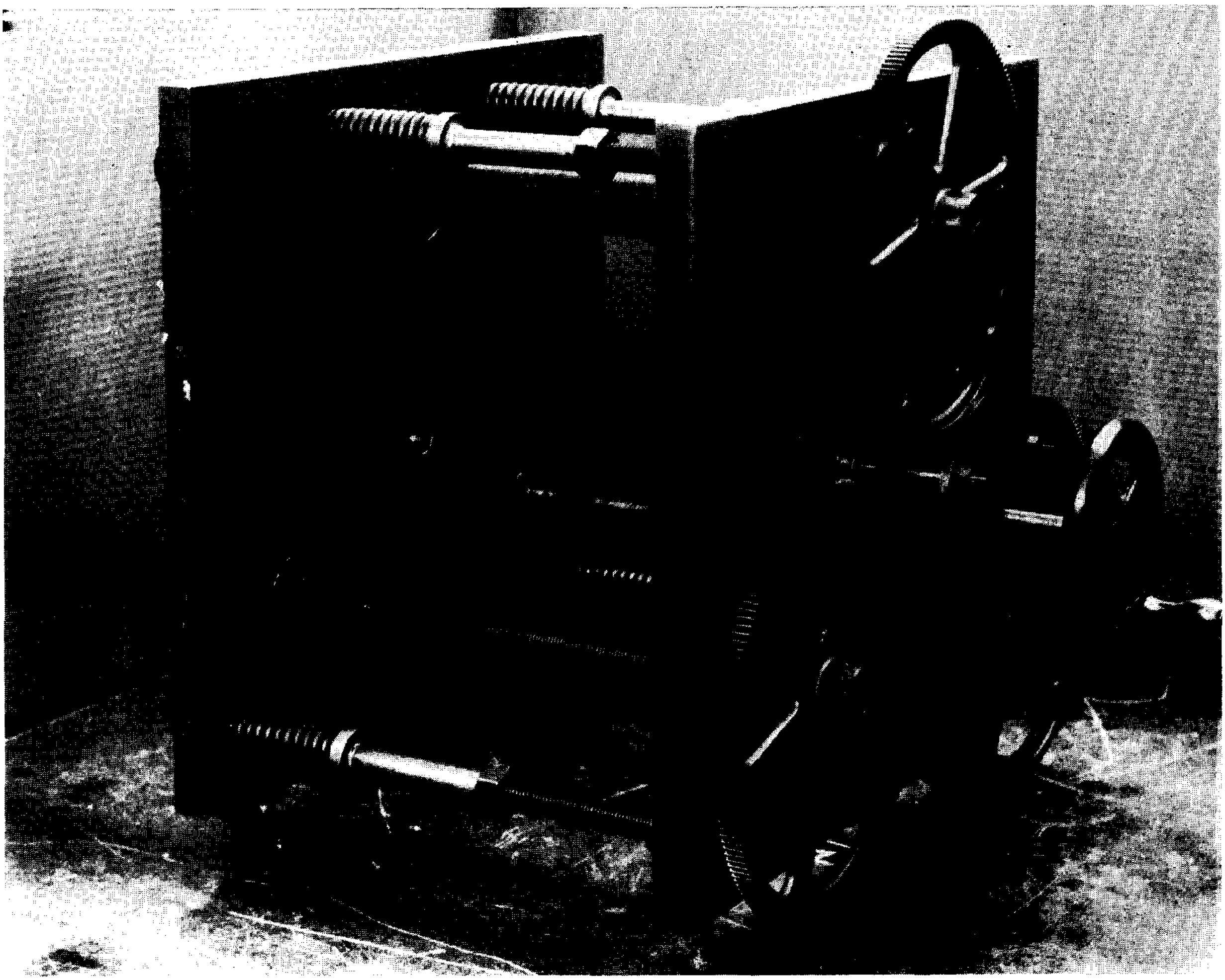

FIG. 2. Testing machine No.1: controlled shortening loading.

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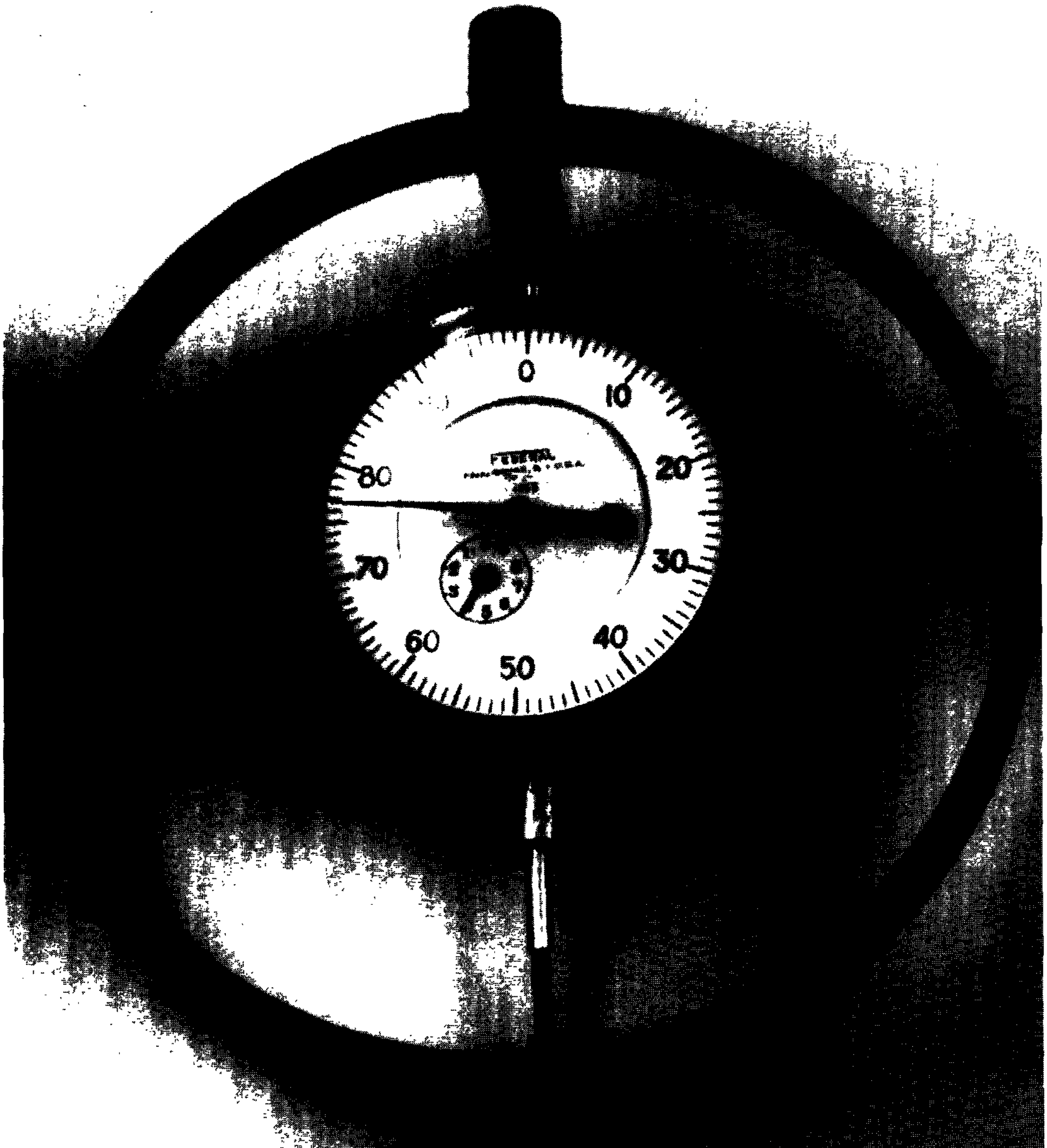

FIG. 3. Testing machine No. 2: load ring.

really very far from rigid during the buckling process. As the high-speed pictures in this report show, the important part of the buckling occurs in about 1/1000 of a second. A rough calculation reveals that the end cap of the cylinder would move about 10^{-5} in. for a free falling beam which is the lever in the testing machine. However, the deflection necessary for dead weight loading is more than one order of magnitude greater than this.

Several tests on cylindrical shells were also carried out by Horton, Johnson and Hoff [8]. Four of these tests were conducted in testing machines that the spring constant could be measured with some reliability. The authors drew no definite conclusions from the tests. The data points have more scatter but are not in disagreement with the experimental data presented in this report.

EXPERIMENTAL DATA

Experimental data on the buckling of electroformed cylindrical shells has been obtained during the past several years at GALCIT.* These shells are electrodeposited on wax mandrals. The resulting test specimens have thickness variations of the order of ± 2 per cent, and have geometrical tolerances of the order of $\pm 1/2$ the thickness. The initial deformation is measured after mounting the shell in the testing apparatus. During the testing of these shells three different types of loading systems have been used. Each of these will be described briefly.

Testing machine No. 1

The first type of testing machine was used in the initial phase of all the experimental work and is shown in Fig. 2. This machine was constructed as a controlled displacement type of machine. The loading is accomplished by the three load screws which have 40 threads per inch. The screws can be turned individually for adjusting the load distribution or simultaneously for increasing the total load. The load is applied to the test cylinder through an intermediate cylinder [10]. This cylinder is instrumented with strain gages which give the load distribution on the test cylinder and the total load applied.

The stiffness of the machine was determined by measuring the deflection of the screws, bearings, and end plates under load. In this manner the machine stiffness, K, was determined to be 500,000 lb/in. In addition, the intermediate cylinder with the strain gages has a stiffness of 2,000,000 lb/in. The resultant stiffness of the whole loading system then becomes 400,000 lb/in if an intermediate cylinder is used on one end and 333,000 lb/in if an intermediate cylinder is used on both ends.

Testing machine No. 2

This testing machine consisted of a simple loading ring as shown in Fig. 3. The shell was loaded using this ring in a 300,000 lb testing machine. Assuming the testing machine is very rigid as compared to the load ring, the spring constant of this loading system is 25,000 lb/in.

Testing machine No. 3

This type of testing machine applies the load by means of a pressure diaphragm similar to the method of loading used in a flutter test at GALCIT [11]. The pressure is fed to a

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flexible tube that is constrained by the loading fixture and the end ring of the shell. A drawing of the loading fixture is shown in Fig. 4. The loading applied by the pressure diaphragm appears to be dead weight except for the stiffness of the tube. This stiffness was measured to be about 600 lb/in.

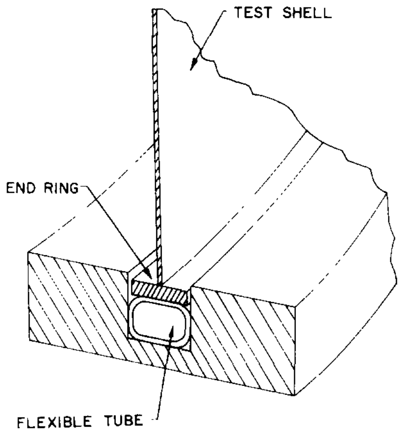

FIG. 4. Testing machine No. 3: pressure loading.

Buckling data

A total of eight shells suitable for comparison have been tested in these three machines. Table 1 shows the dimensions of the shell, the spring constant defined to be

$$K_s = 2\pi RtE/L$$

and the type of testing device. The shells do not have the same dimensions, but they all fall into the long-shell range since the value of L^2/Rt is greater than 2300 for each shell. However, previous experimental data on the buckling of cylindrical shells under axial compression show that the buckling load has an R/t dependence. A correlation of the data [12] shows that the maximum difference of the buckling load of the shells due to geometric effects should be about 6 per cent of σ_{cl}. Since the difference to be expected for the effect of testing machine on the buckling load is of the order of 30 per cent of σ_{cl}, this difference should be easily detectable.

The shells were buckled using the three different loading methods previously described. However, all the shells except No. 6 were mounted in the same manner. In order to determine the load and the load distribution, the shell is mounted into a load measuring

TABLE 1. EXPERIMENTAL DATA

Shell	L (in.)	$t \times 10^3$ (in.)	R/t	$E \times 10^{-6}$ (lb/in^2)	$K_s \times 10^{-5}$ (lb/in)	Testing machine	K/K_s	σ/σ_{cl}
1	7·0	5·28	760	16·0	3·03	1	1·10	0·618
2	8·0	4·41	910	16·0	2·22	1	1·50	0·656
3	10·0	4·69	850	16·7	1·97	1	2·03	0·687
4	10·0	5·12	790	15·8	2·04	1	1·96	0·684
5	8·0	3·91	1020	16·0	1·96	2	$1·28 \times 10^{-1}$	0·688
6	8·0	3·46	1160	16·6	1·79	3	$3·35 \times 10^{-3}$	0·674
7	8·1	4·16	960	17·2	2·22	3	$2·70 \times 10^{-3}$	0·612
8	6·9	4·18	960	16·7	2·53	3	$2·37 \times 10^{-3}$	0·655

cylinder [10]. Since all shells were mounted into the same load cell, the shell boundary conditions are the same. Shell No. 6 was mounted into very rigid end plates. However, the load for this shell is not significantly greater than the other shells tested with this machine. This is in agreement with the results recently found for the buckling of shells with elastic end supports [13,14]. These results establish that all of the shells tested had boundary supports that should be rigid enough to provide clamped end conditions.

The buckling stress or collapse stress is shown in the table and plotted vs. the ratio of test shell to testing device stiffness in Fig. 5. The data has a total spread of about 7 per cent of σ_{cl}. However, the figure shows that the buckling load is not dependent on the stiffness of the loading device over three orders of magnitude.

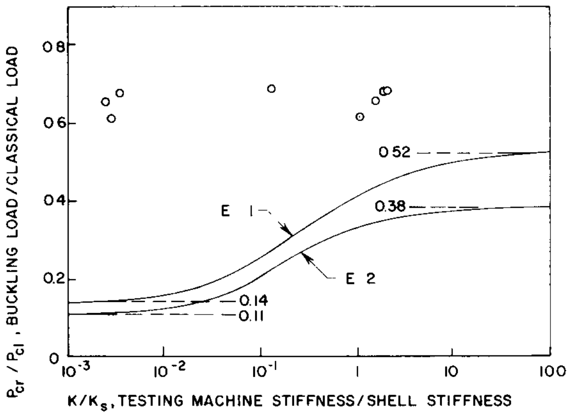

FIG. 5. Buckling load vs. testing machine stiffness: experimental results and results for energy criteria.

ENERGY CRITERION

In 1942, Tsien [2] postulated what has been commonly called an energy criterion of buckling. As a revision of his original criterion he proposed a second criterion in 1947 [3]. These energy criteria are based upon the assumption that under ordinary conditions the shell should jump from the prebuckled to the postbuckled state. This jump will occur if a postbuckled state exists that is available to the shell testing machine system. The necessary energy for the transition between the two states is assumed to come from the local environment.

These energy criteria have been widely discussed and have been shown to be inconsistent with some experimental results. In particular, in a series of experiments by Fung and Kaplan [15] the energy criterion first proposed by Tsien was shown to give results disagreeing with experiment. The experiment consisted of a series of tests on low arches where the testing method was the same for a wide range of arch geometry. The purpose of this following calculation is to find the buckling load as given by an energy criterion for a range of testing machine stiffness. This calculation will then be compared with the experimental data for shells of roughly the same geometry.

Tsien's first energy criterion

The first criterion, as proposed by Tsien, stated that the shell should jump from the unbuckled state to the buckled state if the potential energy of the two states are equal.* The calculation depends upon the type of the testing machine and the postbuckled equilibrium states of the shell. Tsien did this calculation for the dead weight and rigid loading cases, using the load deflection relation as calculated by von Kármán and Tsien [16] for a cylindrical shell. However, the energy load as defined above can easily be calculated for any loading system once the load vs. deflection relation for the shell and the loading system is known.

The potential energy of the system as shown in Fig. 6 is made up of the strain energy stored in the shell and the strain energy stored in the loading spring. In this figure the testing machine is idealized as a rigid loading head moving through a displacement Δ and an elastic loading spring. Assuming the loading spring is linear with spring constant K lb/in this energy can be calculated to be

$$PE = \tfrac{1}{2}K(\Delta-\delta)^2 + U_s. \tag{1}$$

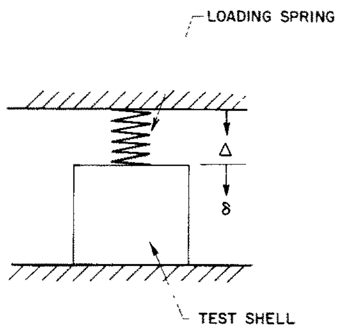

FIG. 6. Idealization of loading system.

The strain energy of the shell, U_s, can be calculated from a load vs. deflection (p vs. δ) graph of the shell as follows:

$$U_s = \int p\,\mathrm{d}\delta. \tag{2}$$

Using the p vs. δ graph for a cylindrical shell calculated by Almroth† [9] and shown in Fig. 1, the strain energy can be calculated and is shown in Fig. 7. In these figures the following nondimensional quantities have been introduced.

$$\bar{U}_s = U_s \quad \frac{\pi}{4}\left(\frac{t}{R}\right)^2 ELRt$$

$$\bar{\sigma} = \sigma/\sigma_{cl} = \sigma \Big/ \frac{Et}{R\sqrt{[3(1-\nu^2)]}}, \qquad \bar{\delta} = \delta/\delta_{cl} = \delta \Big/ \frac{tL}{R\sqrt{[3(1-\nu^2)]}} \tag{3}$$

* This buckling load was actually introduced by Friedrichs [17] and called the "intermediate buckling load". However, it is almost always referred to as Tsien's energy load.

† There is agreement upon the general shape of this curve for a cylindrical shell under axial compression, but exact values depend upon the approximations made in obtaining the solution [18].

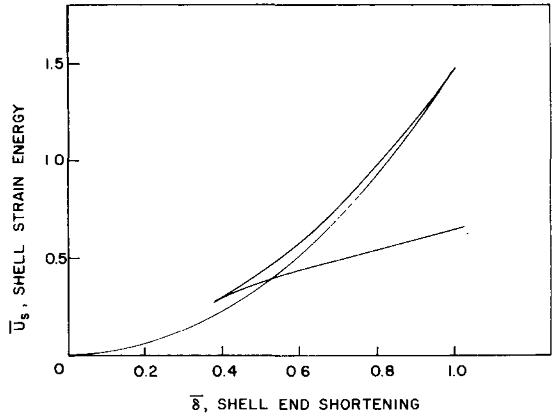

FIG. 7. Strain energy for a cylindrical shell under axial compression.

where $\sigma = p/2\pi Rt$ and Poisson's ratio has been taken to be $\frac{1}{3}$. The strain energy of the shell could also be calculated by using the potential energy plot calculated by Almroth. This calculation is performed as follows:

$$U_s = PE_s + p\delta. \tag{4}$$

In nondimensional form this equation becomes

$$\bar{U}_s = \overline{PE}_s + 3\bar{\sigma}\bar{\delta} \tag{5}$$

where

$$\overline{PE}_s = PE_s \Big/ \frac{\pi}{4}\left(\frac{t}{R}\right)^2 ELRt. \tag{6}$$

Using either equation (2) or (4) the potential energy of the whole system can be calculated for any linear testing machine using the following equations.

$$\overline{PE} = \bar{U}_s + \frac{3}{2}\frac{\bar{\sigma}^2}{\bar{K}} \tag{7}$$

or

$$\overline{PE} = \overline{PE}_s + 3\bar{\sigma}\bar{\delta} + \frac{3}{2}\frac{\bar{\sigma}^2}{\bar{K}} \tag{8}$$

where

$$\bar{K} = K/K_s = K \Big/ \frac{2\pi RtE}{L}. \tag{9}$$

These calculations have been carried out for a range of K from 10^{-3} to 10^2 and a buckling load determined based upon the first energy criterion. The result of this calculation is shown in Fig. 5 and labeled E1.

Tsien's second energy criterion

The next criterion of buckling that Tsien [3] proposed, was that the shell should jump from the unbuckled state to the buckled state whenever there is an available stable equilibrium position in the post-buckled state. However, the availability of such a stable position depends not only on the characteristics of the shell but also on the loading system. If the machine is a dead-weight type of machine, then the jump must be made under constant load and if the machine is a rigid one, the jump must be made under constant displacement. Any linear elastic machine will fall between these two extremes. The load at which this jump will occur can be found from the load deflection curve for the shell by simply drawing the inverse slope of the testing machine on the graph and finding the minimum load for which this line intersects the postbuckled range of the shell. The end result of the calculation is shown in Fig. 5 and labeled E2. Again, Almroth's load vs. deflection relation has been used for this calculation.

CONCLUSION

The results of the experimental work and the calculated energy loads from the two energy criteria are compared in Fig. 5. The variation of the energy load with testing machine stiffness is similar for both criteria but is always lower for the second. This is due to the fact that, for the second criterion, the system is allowed to jump to a state of higher potential energy. The comparison with the experimental points shows that an energy criterion of this type does not properly predict the dependence of the buckling load on the testing machine stiffness.

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Résumé—Une série d'expériences a été exécutée sur des coquilles formées par galvanoplastie sous compression axiale pour établir l'effet de la raideur de l'appareil d'essai sur la charge de fléchissement. L'effet de l'appareil d'essai a aussi été calculé en utilisant les critères de Tsien. Il est démontré que les charges d'énergie calculées dépendent fortement de l'appareil d'essai tandis que les données expérimentales sont virtuellement indépendantes de la raideur et l'appareil d'essai.

Zusammenfassung—Eine Reihe von Versuchen wurde an galvanisch geformten Zylinderschalen unter Längsdruck unternommen, um den Einfluss der Prüfmaschinenstarrheit auf die Knicklast zu bestimmen. Der Einfluss der Prüfmaschine wurde auch bei Anwendung der Tsien'schen Kriterien berechnet. Es wird gezeigt, dass die berechneten Energien stark von der Maschine abhängen, während die Versuchsresultate von der Maschine unabhängig sind.

Абстракт—Проведено несколько серий экспериментов для построенных с помощью электролиза цилиндрических оболочек, подверженных осевому сжатию для определения эффекта жесткости испытательной машины на величину нагрузки выпучивания. Подсчитано также эффект испытательной машины, используя критерия Цзяна. Оказывается, что определенные энергические нагрузки находятся в большой зависимости от испытательной машины, тогда как экспериментальные результаты действительно не зависят от жесткости испытательной машины.